RECONSTRUCTING QUASIMORPHISMS FROM ASSOCIATED PARTIAL ORDERS AND A QUESTION OF POLTEROVICH

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ABSTRACT. We show that every continuous homogeneous quasimorphism on a finite-dimensional 1-connected simple Lie group arises as the relative growth of some continuous bi-invariant partial order on that group. More generally we show, that an arbitrary homogeneous quasimorphism can be reconstructed as the relative growth of a partial order subject to a certain sandwich condition. This provides a link between invariant orders and bounded cohomology and allows the concrete computation of relative growth for finite dimensional simple Lie groups as well as certain infinite-dimensional Lie groups arising from symplectic geometry.

1. INTRODUCTION

In this article we observe a new relation between two different well-known structures on Lie groups. The one side of our correspondence is formed by continuous invariant partial orders. Here a partial order $\leq$ on a topological group G is called *invariant* (or bi-invariant), if for all $g, h, k \in G$ the relation $g \leq h$ implies both $kg \leq kh$ and $gk \leq hk$. This means that the associated *order semigroup*

$$G^+ := \{g \in G \,|\, g \geq e\}$$

is a conjugation-invariant pointed (i.e. $G^+ \cap (G^+)^{-1} = \{e\}$) monoid. Then $\leq$ is called *continuous* if G^+ is closed in G and locally topologically generated (i.e. for every identity neighbourhood U in G the intersection $U \cap G^+$ generates a dense subsemigroup of G^+). Such orders will be related to continuous homogeneous quasimorphism, i.e. continuous maps $f : G \to \mathbb{R}$ satisying $f(g^n) = nf(g)$ for all $g \in G$ and $n \in \mathbb{Z}$, for which the function $f(gh) - f(g) - f(h)$ is bounded on G^2. Both sides of the correspondence individually are well-studied; for finite-dimensional simple Lie groups there are classifications of both (see [9] for invariant orderings and [4] for quasimorphisms). An immediate consequence of these classification results is the following proposition:

Proposition 1.1. *For a finite-dimensional 1-connected simple Lie group G the following are equivalent:*

(i) *There exists a non-trivial continuous invariant partial order $\leq$ on G.*

(ii) *There exists a non-zero continuous homogeneous quasimorphism on G.*

The main result of this article states that we can actually use the continuous invariant partial orders to construct the corresponding continuous homogeneous quasimorphisms explictly. For this we use the machinery of relative growth as introduced in [6]: Given any invariant partial order $\leq$ on a group G we define the associated set of *dominants* in G to be

$$G^{++} = \{g \in G^+ \setminus \{e\} \mid \forall h \in G \exists n \in \mathbb{N} : g^n \geq h\}.$$

We call an invariant order *admissible* if $G^{++} \neq \emptyset$. For a fixed dominant element $g \in G^{++}$ we define the *relative growth*

$$\gamma(g, \cdot) : G \to \mathbb{R}$$

by

$$\gamma(g, h) = \lim_{n \to \infty} \frac{\min\{p \in \mathbb{Z} \mid g^p \geq h^n\}}{n}.$$

Then we provide the following explicit correspondence:

Theorem 1.2. *Let $f : G \to \mathbb{R}$ be a non-zero homogeneous quasimorphism on a 1-connected simple Lie group G. Then there exists a continuous admissible partial order $\leq$ on G such that for any $g \in G^{++}$ and $h \in G$ the relative growth is given by*

$$\gamma(g, h) = \frac{f(h)}{f(g)}.$$

To the best of our knowledge this is the first results which provides a correspondence between invariant order structures and quasimorphisms. In modern language, the two sides of the correspondence are given by Lie semigroups [10, 19] and continuous bounded cohomology classes [18, 4], respectively. In fact, the study of continuous invariant orders reduces to the more classical subject of invariant cones in Lie algebras. Interest in such invariant cones first arose in the context of infinite-dimensional representation theory and mathematical physics (in particular, general relativity) [26, 21, 22]. By now Lie semigroups have found applications in areas as diverse as logic and geometric control theory (see [12] for a historical overview). On the other hand, bounded cohomology in general and quasimorphisms in particular are an indispensable tool in modern geometric group theory. Some articles of particular relevance to the present work are [1, 2, 4, 5]. We hope that the present work will initiate more interaction between these two rich and traditional areas of topological group theory. We would like to point out that the theory of relative growth was originally developed in [6] in a completely different context, namely the study of infinite-dimensional Lie groups arising from problems in contact and symplectic geometry.

In order to motivate the first step in the proof of Theorem 1.2, we consider a purely algebraic variant of that theorem, which applies to general groups G and arbitrary homogeneous quasimorphism $f : G \to \mathbb{R}$. For such a pair (G, f) one can always construct an invariant partial order $\leq_f$ on G by demanding that

$$g <_f h \Leftrightarrow f(g) < f(h) + D(f),$$

where

$$D(f) := \sup_{g,h \in G} (f(gh) - f(g) - f(h))$$

denotes the *defect* of f. Using the theory of relative growth we can show that $\leq_f$ actually determines f up to a positive multiplicative constant. Indeed, we have:

Proposition 1.3. *Let G be a group, $f : G \to \mathbb{R}$ a homogeneous quasimorphism and $\leq_f$ as above. Then $\leq_f$ is admissible, and for any $g \in G^{++}$ and $h \in G$ the corresponding relative growth is given by*

$$\gamma(g,h) = \frac{f(h)}{f(g)}.$$

In particular, up to positive multiple every quasimorphism arises as the relative growth of some partial order with respect to any dominant.

Proposition 1.3 is an interesting observation in its own right, but it does *not* imply Theorem 1.2 directly. The reason is that in the situation of Theorem 1.2 the order $\leq_f$ associated with f will be badly behaved. For example, G^+ will not be connected, and consequently the order cannot be continuous. We will thus need a stronger version of Proposition 1.3. Namely, we will show in Proposition 3.2 below that a homogeneous quasimorphism f can be recovered as relative growth from *any* invariant partial order, which agrees with $\leq_f$ up to some bounded error. In a second step we have to obtain explicit descriptions of all possible continuous quasimorphisms and sufficiently many continuous partial orders on 1-connected simple Lie groups. The main work then lies in the third and final step, where we use the results of Step 2 in order to verify that every continuous quasimorphism is related to one of the continuous orders. While the first step uses only elementary methods, the other two steps depend on an in depth understanding of the fine structure of the Lie groups under consideration.

This article is organized as follows: In Section 2 we recall the structure of those 1-connected simple Lie groups, which admit continuous invariant partial orders. These turn out to be Hermitian. We obtain explicit descriptions of both continuous homogeneous quasimorphisms and continuous invariant partial orders on such groups. In Section 3 we prove Theorem 1.2 along the lines explained above. We first provide the necessary generalization of Proposition 1.3 and use it to reduce the statement of the theorem to an estimate on the values of the quasimorphism in question. This estimate willl be established separately for the contribution coming from a maximal compact subgroup and a complementary non-compact contribution. In a final subsection we indicate briefly how to generalize our results beyond the simply-connected case. The concluding Section 4 discusses various applications and extensions of the main result. Following [6] we introduce the notion of an order space, which is a certain metric space associated to an ordered group. We explain how our main results allow one to compute the order space of 1-connected simple Hermitian Lie group for suitable orderings. This answers in particular a question of Polterovich, which was the starting point for the present article. We then discuss possible extensions of

our results to infinite-dimensional Lie groups arising in symplectic geometry. Again, we are able to compute certain order spaces, and the results are in strong contrast to existing results about similar infinite-dimensional groups in the symplectic context.

Convention 1.4. In order to avoid tedious repetitions, throughout the body of this article all homogeneous quasimorphisms are assumed to be continuous. (Note that in fact any homogeneous quasimorphism on a finite-dimensional simple Lie group is automatically continuous [25].)

2. Quasimorphisms and partial orders on Hermitian Lie groups

2.1. **The structure of Hermitian Lie groups.** Let G be a 1-connected simple real Lie group and $G_0 := \mathrm{Ad}(G)$ so that $\pi : G \to G_0$ is a universal covering map. Fix a maximal compact subgroup $K_0 \subset G_0$ and define $K := \pi^{-1}(K_0)$. Then $\mathcal{X} := G/K = G_0/K_0$ is a symmetric space and G is called Hermitian if $\Omega^2(\mathcal{X})^G \neq \{0\}$. In this case, actually, $\Omega^2(\mathcal{X})^G \cong \mathbb{R}$. It was already known to Vinberg [26] that among simple Lie groups only the Hermitian ones can admit continuous invariant partial orders. We will thus focus on such Lie groups in the sequel. In order to fix our notation we briefly recall the structure theory of 1-connected simple Hermitian Lie groups. For more details the reader is asked to consult [14], [8, Chapter III] and (regarding compact Lie groups) [13, Chapter IV]. Throughout, the Lie algebra of a Lie group is denoted by the corresponding small gothic letter; a subscript $\mathbb{C}$ indicates complexification.

- $\mathfrak{k}$ decomposes as $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}'$, where $\mathfrak{z}(\mathfrak{k})$ denotes the 1-dimensional center of $\mathfrak{k}$ and $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ its semisimple part. This induces global decompositions $K = Z(K) \times K'$ and $K_0 = Z(K_0) \times K_0'$. Here K' is a finite covering of K_0', hence compact, while $Z(K) \cong \mathbb{R}$. In particular, both K' and $Z(K)$ and hence K are amenable.
- There exists a Cartan subgroup H of G with $Z(K) \subset H \subset K$. We fix such a Cartan subgroup once and for all.
- Denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{p}$ with respect to the Killing form so that $\mathfrak{g} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}' \oplus \mathfrak{p}$. Identify $\mathfrak{p}$ with the tangent space of symmetric space $\mathcal{X}$ of G at the basepoint eK. There are two choices for the invariant complex structure of $\mathcal{X}$, and we fix one of them. After this choice, there exists a unique $J \in \mathfrak{z}(\mathfrak{k})$ such that $\mathrm{ad}(J)|_{\mathfrak{p}}$ defines the restriction of the chosen complex structure to $\mathfrak{p}$.

- Denote by $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ the roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. Choose a positive system $\triangle^+ \subset \triangle$ in such a way that for all α in the set $\triangle_n^+$ of non-compact positive roots the relation $\alpha(iJ) = 1$ holds. Fix a maximal system of strongly orthogonal roots $\triangle_n^{++} \subset \triangle_n^+$.
- The *compact Weyl group* (associated to our choice of compact Cartan H) is defined by $W_c := N_K(H)/Z_K(H)$. This acts on H by conjugation and thus on $\mathfrak{h}$ via the adjoint action and $\mathfrak{h}^*$ via the coadjoint action. We denote by $(\mathfrak{h}^*)^{W_c}$ and $(\mathfrak{h})^{W_c}$ the sets of W_c-invariants in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. Our choice of non-compact root ensures that $\triangle_n^+$ is invariant under W_c.
- Given $\alpha \in \triangle$ choose root vectors $E_{\pm\alpha} \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$ such that

$$i(E_\alpha + E_{-\alpha}),\ E_\alpha - E_{-\alpha} \in \mathfrak{k} + i\mathfrak{p}, \quad \alpha([E_\alpha, E_{-\alpha}]) = 2.$$

  Define $h_\alpha := -i[E_\alpha, E_{-\alpha}] \in \mathfrak{h}$, $X_\alpha := E_\alpha + E_{-\alpha}$ and $Y_\alpha := -i(E_\alpha - E_{-\alpha})$. Denote by $\mathfrak{a}$ the span of the X_α for $\alpha \in \triangle_n^{++}$ (which is a maximal abelian subalgebra of $\mathfrak{p}$) and by A the associated analytic subgroup of G.
- We use the isomorphism $\ker(\pi) = C(G) \cong \pi_1(G_0)$ to identify $\pi_1(G_0)$ with a subgroup of G. We observe that $\pi_1(G_0) \cong \pi_1(K_0)$ is actually a subgroup of K.

2.2. **Continuous homogeneous quasimorphisms.** We keep the notations introduced in the last subsection; in particular, G is a 1-connected simple Hermitian Lie group. We describe homogeneous quasimorphisms[1] on G. For background on continuous bounded cohomology and quasimorphisms we refer the reader to [18], [2], [1]. Homogeneous quasimorphisms on Hermitian Lie groups are discussed in detail in [5]. In particular we deduce from [5, Prop. 7.8]:

Lemma 2.1. *There exists a unique homogeneous quasimorphism $\mu_G : G \to \mathbb{R}$ satisfying $d_e\mu_G(J) = 1$. Any homogeneous quasimorphism on G is a multiple of μ_G.*

The following observation will be needed later:

Proposition 2.2. *Let μ_G be a homogeneous quasimorphism on G. Then for all $p \in \exp(\mathfrak{p})$ we have $\mu_G(p) = 0$.*

Proof. Since A is amenable, the restriction $\mu_G|_A$ is a homomorphism. Since homogeneous quasimorphisms are conjugation-invariant, its differential is invariant under $X \mapsto \exp(\mathrm{ad}(\pi J))(X) = -X$. This shows that $\mu_G|_A$ is trivial, and the proposition follows by using conjugation-invariance once more. $\square$

By conjugation invariance the restriction $\mu_G|_K$ is uniquely determined by $\mu_G|_H$; the latter is a homomorphism, which can be determined explicitly. For this the key observation is the following lemma:

Lemma 2.3. *With notation as above we have $\dim(\mathfrak{h}^*)^{W_c} = \dim(\mathfrak{h})^{W_c} = 1$.*

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[1] Recall our convention that homogeneous quasimorphisms are assumed continuous.

Proof. Decompose $\mathfrak{h}$ into irreducibles W_c-modules. As a first step let $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{k}'$ so that $\mathfrak{h} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{h}'$. Then $\mathfrak{h}'$ is a maximal torus in the compact semisimple Lie algebra $\mathfrak{k}'$, i.e. $\mathfrak{h}'_{\mathbb{C}}$ is a maximal torus in the complex semisimple Lie algebra $\mathfrak{k}'_{\mathbb{C}}$ and W_c is the Weyl group associated to the pair $(\mathfrak{k}'_{\mathbb{C}}, \mathfrak{h}'_{\mathbb{C}})$. In particular, the action of W_c on $\mathfrak{z}(\mathfrak{k})$ is trivial, while $\mathfrak{h}'$ decomposes into irreducible modules corresponding to the simple subalgebras of $\mathfrak{k}'$. Each of these modules has dimension ≥ 3 and is thus non-trivial. Thus, $\dim(\mathfrak{h})^{W_c} = 1$. Now fix a non-degenerate invariant bilinear form on $\mathfrak{k}$. The restriction of this form to $\mathfrak{h}$ can then be used to identify $\mathfrak{h}$ and $\mathfrak{h}^*$ as W_c-modules. This yields $\dim(\mathfrak{h}^*)^{W_c} = \dim(\mathfrak{h})^{W_c}$. $\qquad\square$

Now we deduce easily:

Proposition 2.4. *Let notation be as above. Then for all $X \in \mathfrak{h}$ we have*

$$(1) \qquad d\mu_G(X) = \frac{1}{|\triangle_n^+|} \sum_{\alpha \in \triangle_n^+} \alpha(iX).$$

Proof. Both $d\mu_G|_H$ and $\sum \alpha(i\cdot)$ define elements in $(\mathfrak{h}^*)^{W_c}$, hence are proportional. The proportionality constant can be computed by evaluating at J. $\qquad\square$

2.3. **Continuous partial orders.** In this section we describe continuous partial orders on a 1-connected simple real Lie group G. We keep the notation of the last two sections. Associated with any such order $\leq$ is a closed, topologically locally generated order semigroup G^+. By results of Neeb [19], G^+ is a Lie semigroup. This means that the Lie wedge

$$C^+ := \mathbf{L}(G^+) = \{X \in \mathfrak{g} \,|\, \forall t > 0 : \exp(tX) \in G^+\}$$

generates G^+ infinitesimally. In particular, $\leq$ is uniquely determined by Ad-invariant closed, pointed generating cone C^+. Such cones have been classified in [26], [21] and [22], and it is well-known that any such cone is determined uniquely by its intersection with $\mathfrak{h}$. Moreover, any such cone contains either J or $-J$ and the corresponding partial order will be called *positive* or *negative* accordingly. Now the aforementioned classification results imply:

Lemma 2.5. *There exists a unique finest positive partial order $\leq$ on G. If $G_{\max}^+$ denotes the corresponding order semigroup, then*

$$(2) \qquad c_{\max}^+ := \mathbf{L}(G_{\max}^+) \cap \mathfrak{h} = \{X \in \mathfrak{h} \,|\, \forall \alpha \in \triangle_n^+ : \alpha(iX) \geq 0\}.$$

3. Realizing quasimorphisms as relative growth

3.1. **The sandwich condition.** We have observed in the introduction that a quasimorphism can be reconstructed as relative growth from the associated partial order. A more general condition allowing for such a reconstruction is the following:

Definition 3.1. A homogeneous quasimorphism f on a topological group G is said to *sandwich* an invariant partial order $\leq$ if there exist constants $C_1, C_2 \in \mathbb{R}$ such that

$$Q_f^+(C_1) \subset G^+ \subset Q_f^+(C_2),$$

where for $C \in \mathbb{R}$ the *superlevel set* $Q_f^+(C)$ is given by.

$$Q_f^+(C) := \{g \in G \mid f(g) \geq C\}$$

Using the homogeneity of f it is easy to see that the sandwich condition is equivalent to the existence of a constant $C_1 > 0$ with

(3) $$Q_f^+(C_1) \subset G^{++} \subset G^+ \subset Q_f^+(0).$$

The following generalization of Proposition 1.3 will be at the heart of our proof of Theorem 1.2. We therefore decided to provide a full proof, although the argument is elementary:

Proposition 3.2. *Suppose that $(G, \leq)$ is ordered group and that $f : G \to \mathbb{R}$ is a non-trivial homogeneous quasi-morphism. If f sandwiches $\leq$, then $\leq$ is admissible and for all $g \in G^{++}$, $h \in G$ we have*

$$\gamma(g, h) = \frac{f(h)}{f(g)}.$$

Proof. Let $g \in G^{++}$ and $h \in G$. Define $T_n(g, h) := \{p \in \mathbb{Z} \mid g^p \geq h^n\}$ and $\gamma_n(g, h) = \inf T_n(g, h)$ so that

(4) $$\gamma(g, h) = \lim_{n \to \infty} \frac{\gamma_n(g, h)}{n}.$$

Choose a constant $C_1 > 0$ such that (3) holds. Since f is non-trivial, $Q_f^+(C_1)$ is non-empty, and thus $\leq$ is admissible. We claim that any integer p_n satisfying

$$p_n \geq \frac{nf(h) + C_1 + D(f)}{f(g)}$$

also satisfies $p_n \in T_n(g, h)$. (Such a p_n exists since $f(g) \neq 0$ for any dominant g.) Indeed, we have

$$f(g^{p_n}h^{-n}) \geq p_n f(g) - nf(h) - D(f) \geq (nf(h) + C_1 + D(f)) - nf(h) - D(f) = C_1,$$

hence $g^{p_n}h^{-n} \in G^+$, which implies $g^{p_n} \geq h^n$ as claimed. In particular $\gamma_n(g, h) \leq p_n$ and choosing p_n minimal possible we obtain

(5) $$\frac{\gamma_n(g, h)}{n} \leq \frac{f(h)}{f(g)} + \frac{C_1 + D(f) + f(g)}{nf(g)}.$$

Now suppose $p \in \mathbb{Z}$ satisfies

$$p < \frac{nf(h) - D(f)}{f(g)}.$$

Then

$$f(g^p h^{-n}) \leq pf(g) - nf(h) + D(f) < (nf(h) - D(f)) - nf(h) + D(f) = 0.$$

Thus $g^p h^{-n} \notin G^+$ and thus $p \notin T_n(g, h)$. Consequently,

(6) $$\frac{\gamma_n(g, h)}{n} \geq \frac{f(h)}{f(g)} - \frac{D(f)}{nf(g)}.$$

Combining (5) and (6) and passing to the limit $n \to \infty$ we obtain the proposition. $\square$

By means of Proposition 3.2 we can reduce the proof of Theorem 1.2 to the following observation:

Lemma 3.3. *For every* 1-*connected Hermitian simple Lie group* G, *the quasimorphism* μ_G *sandwiches the maximal continuous positive ordering* $\leq$.

The remainder of this section is devoted to the proof of Lemma 3.3 (and hence Theorem 1.2). We will drop the quasimorphism μ_G from the notation thus writing $Q^+(C) := Q^+_{\mu_G}(C)$ for the superlevel sets. With this notation we the have to find constants $C_1, C_2 \in \mathbb{R}$ such that

$$(7) \qquad Q_+(C_1) \subset G^+ \subset Q_+(C_2),$$

We claim that this will follow from the following two lemmata:

Lemma 3.4. *There exist constants* $C'_1, C'_2 \in \mathbb{R}$ *such that*

$$(8) \qquad Q_+(C'_1) \cap K \subset G^+ \cap K \subset Q_+(C'_2) \cap K.$$

Lemma 3.5. *There exists a constant* C_0 *such that for all* $p \in \exp_G(\mathfrak{p}) \subset G$ *there exists* $k(p) \in K$ *with* $|\mu_G(k(p))| \leq C_0$ *and* $k(p)p \geq e$.

Indeed, every $g \in G$ can be written as $g = kp$ with $k \in K$, $p \in \exp_G(\mathfrak{p})$. According to Lemma 3.5 we can choose $k_1 := k(p), k_2 := k(p^{-1})^{-1}$ with

$$k_1 p \geq e \geq k_2 p$$

and $|\mu_G(k_j)| \leq C_0$ for $j = 1, 2$. Now we claim that Lemma 3.4 provides the desired estimate (7) for

$$C_1 := C'_1 + C_0 + D(\mu_G), \quad C_2 := C'_2 - C_0 - D(\mu_G).$$

Indeed, using Lemma 2.2 suppose $\mu(g) \geq C_1$. Then

$$\mu_G(k) \geq \mu_G(kp) - \mu_G(p) - D(\mu_G) = \mu_G(g) - D(\mu_G) \geq C_1 - D(\mu_G)$$
$$\Rightarrow \quad \mu_G(kk_1^{-1}) = \mu_G(k) - \mu_G(k_1) \geq C_1 - D(\mu_G) - C_0 = C'_1$$
$$\Rightarrow \quad kk_1^{-1} \in Q_+(C'_1) \cap K \subset G^+ \cap K$$
$$\Rightarrow \quad k \geq k_1$$
$$\Rightarrow \quad g = kp \geq k_1 p \geq e.$$

Conversely suppose $g \in G^+$. Then

$$g = kp \geq e \geq k_2 p$$
$$\Rightarrow \quad k \geq k_2$$
$$\Rightarrow \quad kk_2^{-1} \in G^+ \cap K \subset Q_+(C'_2) \cap K$$
$$\Rightarrow \quad C'_2 \leq \mu_G(kk_2^{-1}) = \mu_G(k) - \mu_G(k_2) \leq \mu(k) + C_0$$
$$\Rightarrow \quad \mu(k) \geq C'_2 - C_0$$
$$\Rightarrow \quad \mu(g) = \mu(kp) \geq \mu(k) + \mu(p) - D(\mu_G) \geq C'_2 - C_0 - D(\mu_G) = C_2.$$

This proves the claim and reduces Theorem 1.2 to the above two lemmata, whose respective proofs will be the content of the following two subsections.

3.2. **Proof of the main theorem I: The compact contribution.** In this subsection we prove Lemma 3.4. The proof proceeds in two steps, first reducing the problem to the Cartan subgroup H and then comparing the expressions (1) and (2). The first step is an easy consequence of the work of Mittenhuber and Neeb:

Lemma 3.6. *The exponential map* $\exp : \mathfrak{k} \to K$ *is onto and* $k \in K \cap G^+$ *iff there exists* $X \in c_{\max}^+$ *such that* k *is conjugate to* $\exp(X)$.

Proof. Since $Z(K)$ (abelian) and K' (compact) are commuting exponential Lie groups, their product K is exponential. Moreover, $\mathfrak{k}$ is a compact Lie algebra and $G^+ \cap K \subset K$ is an invariant Lie semigroup. Thus [17, Corollary 11] implies $G^+ \cap K = \exp(C_{\max}^+ \cap \mathfrak{k})$. The the lemma follows in view of

$$C_{\max}^+ \cap \mathfrak{k} = \mathrm{Ad}(K)(C_{\max}^+ \cap \mathfrak{h}) = \mathrm{Ad}(K)(c_{\max}^+).$$

$\square$

It thus remains to prove:

Lemma 3.7. *There exist constants* $C_1', C_2' \in \mathbb{R}$ *such that*

$$(9) \qquad Q_+(C_1') \cap H \subset \exp(c_{\max}^+) \subset Q_+(C_2') \cap H.$$

Proof. Let $X \in c_{\max}^+$. In view of (1) and (2) we then have

$$\mu_G(\exp(X)) = d\mu_G(X) = \cdot \frac{1}{|\triangle_n^+|} \sum_{\alpha \in \triangle_n^+} \alpha(iX) \geq 0,$$

which establishes the right inclusion with $C_2' = 0$. For the left inclusion we consider the decomposition $K = Z(K) \times K'$ and define $H' := H \cap K'$. Since H' is compact the kernel Γ of the exponential map $\exp : \mathfrak{h}' \to H'$ possesses a compact fundamental domain D in $\mathfrak{h}'$. Since $\mathfrak{z}(\mathfrak{k}) = \mathbb{R} \cdot J$ and the exponential map $\mathfrak{h} \to H$ is onto, every element $h \in H$ may be written as

$$(10) \qquad h = \exp(tJ + X) \quad (t \in \mathbb{R}, X \in D).$$

Then

$$\mu_G(h) = d\mu_G(tJ + X) = t + r(X),$$

where r is some continuous function on D, hence bounded. This means that $\mu_G(h)$ is large only if t is large, which in turn implies that $tJ + X \in c_{\max}^+$. This yields the left inclusion and finishes the proof. $\square$

3.3. **Proof of the main theorem II: The non-compact contribution.**
The purpose of this subsection is to establish Lemma 3.5, thereby finishing the proof of Theorem 1.2. We will argue by reduction to the case of the universal covering group of $SL_2(\mathbb{R})$, which we denote by $\widetilde{SL_2}(\mathbb{R})$. (This case was treated in [3].) We recall that for any $\alpha \in \triangle_n^{++}$ the bracket relations $[X_\alpha, Y_\alpha] = -2h_\alpha$, $[h_\alpha, X_\alpha] = 2Y_\alpha$ and $[h_\alpha, Y_\alpha] = -2X_\alpha$ hold. (See e.g.[22], where the notation is compatible with ours.) Therefore, the three-dimensional real Lie algebra $\mathfrak{sl}_\alpha$ spanned by $X_\alpha, Y_\alpha, h_\alpha$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ via an isomorphism σ_α given by

$$\sigma_\alpha(X_\alpha) := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_\alpha(Y_\alpha) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_\alpha(h_\alpha) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Denote by $\psi_\alpha : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_\alpha \hookrightarrow \mathfrak{g}$ the inclusion induced by the inverse of this isomorphism. Then ψ_α integrates to a group homomorphism

$$\Psi_\alpha : \widetilde{SL_2}(\mathbb{R}) \to G.$$

In fact, Ψ_α factors through a map $\Psi_\alpha^0 : SL_2(\mathbb{R}) \to G_0$, in particular

$$(11) \qquad \Psi_\alpha(\pi_1(SL_2(\mathbb{R}))) \subset \pi_1(G_0).$$

Indeed, since $SL_2(\mathbb{C})$ is simply-connected the complexification

$$(\psi_\alpha)_\mathbb{C} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}_\mathbb{C}$$

integrates to a map $(\Psi_\alpha)_\mathbb{C} : SL_2(\mathbb{C}) \to (G_0)_\mathbb{C}$. Since G_0 is linear and connected, it coincides with the analytic subgroup of its universal complex-ification with Lie algebra $\mathfrak{g}$ [11, Satz I.6.1 and Satz III.9.24]. Now $(\psi_\alpha)_\mathbb{C}$ maps $\mathfrak{sl}_2(\mathbb{R})$ into $\mathfrak{g}$ and thus $(\Psi_\alpha)_\mathbb{C}$ maps $SL_2(\mathbb{R})$ into G_0. Then the restric-tion of $(\Psi_\alpha)_\mathbb{C}$ provides the desired factorization Ψ_α^0 of Ψ_α. We will now use the maps Ψ_α to reduce our problem to the case of $\widetilde{SL}_2(\mathbb{R})$ by means of the following lemma:

Lemma 3.8. *For each $\alpha \in \triangle_n^{++}$ there exists a continuous admissible partial ordering $\leq_\alpha$ on $\widetilde{SL}_2(\mathbb{R})$ with the following property: If $g \geq_\alpha e$ for some $g \in \widetilde{SL}_2(\mathbb{R})$, then $\Psi_\alpha(g) \in G_{\max}^+$.*

Proof. Denote by $C_{\max}^+$ the Lie wedge of $G_{\max}^+$ and define

$$C_\alpha^+ := \sigma_\alpha(C_{\max}^+ \cap \mathfrak{sl}_\alpha) \subset \mathfrak{sl}_2(\mathbb{R}).$$

Since the kernel of the map

$$\Psi_\alpha : \widetilde{SL}_2(\mathbb{R}) \to \Psi_\alpha(\widetilde{SL}_2(\mathbb{R}))$$

is central, Ψ_α induces an isomorphism $\mathrm{Ad}(\widetilde{SL}_2(\mathbb{R})) \to \mathrm{Ad}(\Psi_\alpha(\widetilde{SL}_2(\mathbb{R})))$. As σ_α is equivariant with respect to these adjoint actions, we deduce that C_α^+ is a Ad-invariant, closed pointed cone in $\mathfrak{sl}_2(\mathbb{R})$. This cone is non-trivial, since $h_\alpha \in C^+$, and thus $\mathfrak{sl}_2(\mathbb{R}) = C_\alpha^+ - C_\alpha^+$, since the right hand side is a non-trivial ideal. This means that C_α^+ is generating. Now there exists only two (mutually inverse) Ad-invariant, closed pointed generating cones in $\mathfrak{sl}_2(\mathbb{R})$, and both are global. This means that there exists a partial order $\leq_\alpha$ on $\widetilde{SL}_2(\mathbb{R})$ with order semigroup $\overline{\langle \exp(C_\alpha^+) \rangle}$. Since

$$\psi_\alpha(C_\alpha^+) = C^+ \cap \mathfrak{sl}_\alpha \subset C^+$$

we have $\Psi_\alpha(\exp(C_\alpha^+)) \subset G^+$, from which the lemma follows. $\square$

To finish our argument we use the following fact about the $\widetilde{SL}_2(\mathbb{R})$-case:

Lemma 3.9. *Given any continuous admissible ordering $\leq$ on $G = \widetilde{SL}_2(\mathbb{R})$ there exists a constant N and an element $z_0 \in \pi_1(\mathrm{SL}_2(\mathbb{R}))$ such that for every $X \in \mathfrak{p}$ there exists $0 \leq n \leq N$ with*

$$z_0^n \exp_G(X) \geq e.$$

Proof. The classification of invariant cones implies that the only continuous admissible orderings on G are those associated with $G_{\max}^+$ and $(G_{\max}^+)^{-1}$. We may thus assume that we are dealing with the maximal positive order. In [3] a different order on G was introduced by dynamical means. It is easy to see that this order is continuous, admissible and positive. By the classification it thus coincides with $G_{\max}^+$. We may therefore freely use the results from [3] in the present context. In particular, we know from [3, Lemma 2.9] (specialized to $n = 1$) that for every $X \in \mathfrak{p}$ there exists a path g_t defining a homotopy class $g = [g_t] \in G$ which satisfies both $g_1 = \exp_{SL_2(\mathbb{R})}(X)$ and $g \geq e$ and has Maslov quasimorphism $\mu_{Maslov}(g) \leq 4\pi$. If we denote $p := \exp_G(X)$, then this means that $g = zp$ for some $z \in \pi_1(\mathrm{SL}_2(\mathbb{R})) \subset G$. If z_0 denotes the

positive generator of $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$, then $z = z_0^n$ for some $n > 0$, and the uniform bound on the Maslov quasimorphism implies the uniform bound on n. □

Now we can finish the proof of Lemma 3.5: Combining Lemma 3.8 and Lemma 3.9 we now choose for every $\alpha \in \triangle_n^{++}$ a constant $N_\alpha \in \mathbb{N}$ and an element $z_{\alpha,0} \in \pi_1(\mathrm{SL}_2(\mathbb{R}))$ such that for every $t_\alpha \in \mathbb{R}$ there exists $0 \le n_\alpha \le N_\alpha$ with

$$z_{\alpha,0}^{n_\alpha} \exp\left(t_\alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) \ge_\alpha e.$$

Define $z_\alpha := \Psi_\alpha(z_{\alpha,0})$. By (11) we have $z_\alpha \in \pi_1(G_0) = Z(G)$. Applying Ψ_α and using Lemma 3.8 we obtain:

$$z_\alpha^{n_\alpha} \exp(t_\alpha X_\alpha) \in G^+.$$

Now, any $a \in A$ is of the form

$$a = \prod_{\alpha \in \triangle^{++}} \exp(t_\alpha X_\alpha)$$

for some $t_\alpha \in \mathbb{R}$, and any $g \in \exp(\mathfrak{p})$ is of the form $g = kak^{-1}$ for some $k \in K$. This implies that for every $g \in \exp(\mathfrak{p})$ we can find $0 \le n_\alpha \le N_\alpha$ such that

$$\left(\prod_{\alpha \in \triangle_n^{++}} z_\alpha^{n_\alpha}\right) \cdot g = k \left(\prod_{\alpha \in \triangle^{++}} z_\alpha^{n_\alpha} \exp(t_\alpha X_\alpha)\right) k^{-1} \in G^+.$$

This implies Lemma 3.5 and finishes the proof of Theorem 1.2.

3.4. **Beyond simple-connectedness.** In the proof of Theorem 1.2 we have always assumed G to be simply-connected. This assumption ensured in particular the existence of a non-zero homogeneous quasimorphism and a non-trivial continuous admissible partial order on G. As far as the former existence question is concerned, it is easy to classify the non-simply connected simple Lie groups $\widehat{G}$ which admit a non-zero homogeneous quasimorphisms. For this we recall that the space of such quasimorphisms is

$$EH_{cb}^2(\widehat{G}; \mathbb{R}) = \ker(H_{cb}^2(\widehat{G}; \mathbb{R}) \to H_c^2(\widehat{G}; \mathbb{R})).$$

Since $\dim H_{cb}^2(\widehat{G}; \mathbb{R}) \le 1$ this is equivalent to $H_{cb}^2(\widehat{G}; \mathbb{R}) \cong \mathbb{R}$ and $H_c^2(\widehat{G}; \mathbb{R}) = 0$. Equivalently, $\widehat{G}$ is Hermitian with finite fundamental group. Denote by $G \to \widehat{G}$ its universal covering. The maximal order on G discussed above may or may not descend to $\widehat{G}$. In the latter case, there exists no continuous admissible ordering on $\widehat{G}$ and no more can be said. We claim that in the former case the induced partial order $\le$ on $\widehat{G}$ is still sandwiched by any non-zero homogeneous quasimorphism of the correct sign. To prove this, denote by $\widehat{K}$ the image of K in $\widehat{G}$ and decompose $\widehat{K} = Z(\widehat{K})\widehat{K}'$ into its center and its semisimple part. Then $Z(K) \to Z(\widehat{K})$ is an isomorphism while $K' \to \widehat{K}'$ is a finite covering. Thus we still have $Z(\widehat{K}) = \{\exp_{\widehat{G}}(tJ) \,|\, t \in \mathbb{R}\}$, and $\widehat{K}'$ is still compact. Therefore the argument in Lemma 3.4 can easily be adapted. The other arguments are essentially Lie algebra arguments, which remain valid without changes, and thus we obtain a proof of Theorem 1.2

also in the case where the group is not simply-connected, as long as the partial order descends to the group in question.

4. Implications and further examples

4.1. **Basic definitions.** Among the initial motivation of Eliashberg and Polterovich to introduce relative growth was the construction of a certain metric G-space out of an admissible ordered group $(G, \leq)$. To explain their construction, let G be a group and $\leq$ an admissible invariant ordering on G. Then the restriction of the relative growth function defines a positive function

$$\gamma : G^{++} \times G^{++} \to \mathbb{R}^{>0},$$

whose symmetrized logarithm

$$d(g, h) := \log \max\{\gamma(g, h), \gamma(h, g)\}$$

yields a pseudo-metric on G^{++}. We refer to the associated metric space as the *order space* of $(G, \leq)$ and denote it by $\mathfrak{X}(G, \leq)$. Note that the conjugation action of G on G^{++} induces an isometric G-action on $\mathfrak{X}(G, \leq)$. In general, it is a difficult problem to compute the order space of an ordered group. However, if the order in question is sanwiched by a homogeneous quasimorphism, then we can apply Proposition 3.2 in order to compute the order space explicitly:

Corollary 4.1. *Suppose that $(G, \leq)$ is an admissible ordered group and that $f : G \to \mathbb{R}$ is a continuous homogeneous quasi-morphism sandwiching $\leq$. Then the map*

$$\iota : \mathfrak{X}(G, \leq) \to \mathbb{R}, \quad [g] \mapsto \log f(g)$$

is an isometry onto its image.

Proof. Let $g, h \in G^{++}$. By Proposition 3.2 we have

$$d([g], [h]) = \max\{\log \gamma(g, h), \log \gamma(h, g)\} = |\log f(g) - \log f(h)|,$$

showing that ι is an isometry. $\qquad\qquad\square$

4.2. **Finite-dimensional examples.** Applying Corollary 4.1 to the case of 1-connected, finite-dimensional simple Lie groups discussed in the main theorem we obtain:

Corollary 4.2. (i) *Let G be a 1-connected simple Hermitian Lie group equipped with its maximal positive order $\leq$. Then there is a surjective isometry*

$$\iota : \mathfrak{X}(G, \leq) \to \mathbb{R}, \quad [g] \mapsto \log \mu_G(g).$$

  (ii) *Denote by $\leq$ the admissible ordering on K obtained by restricting the maximal positive order from G. Then there is still a surjective isometry*

$$\iota : \mathfrak{X}(K, \leq) \to \mathbb{R}, \quad [k] \mapsto \log \mu_G(k).$$

Indeed, Corollary 4.1 applies in view of Lemma 3.3 and Lemma 3.4 respectively, and surjectivity follows from $\mu_G(\exp(tJ)) = t$ in both cases. In fact it is easy to see that every order space of a Lie group necessarily contains a copy of $\mathbb{R}$ as the image of a suitable one-parameter semigroup in G^{++}. In that sense the order spaces of 1-connected simple Hermitian Lie group with respect to the maximal order is as small as possible for a Lie group. Corollary 4.2 answers a question of Polterovich, which was the starting point for the investigations in this paper. The results in Corollary 4.2 should be compared to the case of 1-connected, finite-dimensional *abelian* Lie groups, i.e. finite-dimensional vector spaces.

Example 4.3. Let V be a finite-dimensional vector space (considered as an abelian Lie group under addition) and $C^+ \subset V$ a closed, pointed convex cone with non-empty interior. By [24, Corollary 11.7.1] there exists a weak-$*$-compact subset $\mathcal{A}^*$ of the unit ball V_1^* in V^* such that

$$(12) \qquad C^+ = \{v \in V \mid \forall \alpha \in \mathcal{A}^* : \alpha(v) \geq 0\}.$$

The dominants of the partial order with order semigroup C^+ are given by $C^{++} = \operatorname{Int}(C^+)$. A short computation shows that the pseudo-distance d on C^{++} is given by

$$(13) \qquad d(v, w) = \max_{\alpha \in \mathcal{A}^*} |\log \alpha(v) - \log \alpha(w)|.$$

This is actually a metric on C^{++}, and thus $\mathfrak{X}(V, \leq) = (\operatorname{Int}(C^+), d)$.

Thus in the abelian case, the order space is as large as possible (i.e. the natural map $G^{++} \to \mathfrak{X}(G, \leq)$ is one-to-one), while in the simple case it is as small as possible.

4.3. **Infinite-dimensional examples.** The strong dichotomy between order spaces of finite-dimensional simple and finite-dimensional abelian Lie groups discovered in the last subsection exists also for certain families of infinite-dimensional Lie groups, which we discuss here. For this we return to the original setup, in which relative growth was introduced, namely contact and symplectic geometry. Various infinite-dimensional Lie groups with natural invariant orders arise in this context, and for several classes of such groups the associated order spaces have been studied in [6, 3]. In all these examples the order spaces turn out to be infinite-dimensional. In this subsection we provide an example of a similar geometric flavour, in which the order space fails not only to be infinite-dimensional, but in fact collapses to $\mathbb{R}$. The reason for this collapse is again provided by a homogeneous quasimorphism, which sandwiches the order in question.

In order to explain our example, we introduce the following notation: Denote by (M, ω) a closed symplectic manifold of dimension $2n$. Every smooth, time dependent function $H_t : M \to \mathbb{R}$ gives rise to a smooth vector field, X_H via the pointwise linear equation $dH = -\omega(X_H, \cdot)$. These vector fields are called *Hamiltonian*. The group $G_0 := \operatorname{Ham}(M, \omega)$ of *Hamiltonian motions* is by definition the subgroup of the diffeomorphism group $\operatorname{Diff}(M)$ given by the time-1 maps of the flows generated by the Hamiltonian vector fields. Since ω^n is a volume form on M, G_0 is actually a subgroup of the

volume preserving diffeomorphisms of M. A detailed study of the group G_0 is provided in [23]. Here we just remark that G_0 admits a natural topology and smooth structure, turning it into an infinite-dimensional Lie group. We will be interested in the universal covering G of G_0.

An important problem is the existence and uniqueness problem for Calabi type quasimorphisms on G. For background on this complex of problems see [16, Chapter 10]. Here we recall only some of the most basic definitions in order to fix our notation: Given an open subset $U \subset M$, denote by G_U^0 the group of Hamiltonian diffeomorphisms of M generated by Hamiltonians supported inside U, and observe that the elements of G_U^0 are then automatically compactly supported. On the universal covering G_U of G_U^0 there exists a homomorphism $\mathrm{Cal}_U : G_U \to \mathbb{R}$ called the *Calabi homomorphism* given as follows: If $[f_t] \in G_U$ is represented by a path f_t in G_U^0 generated by a time-dependent Hamiltonian F_t, then

$$\mathrm{Cal}_U([f_t]) = \int_0^1 \int_U F_t \omega^n dt.$$

This homomorphism descends to a homomorphism of G_U^0 if $\omega|_U$ is exact, but not in general. Now let us call an open subset $U \subset M$ *displacable* if there exists $g \in G_0$ such that $gU \cap \overline{U} = \emptyset$. Then we define:

Definition 4.4. A quasimorphism $f : G \to \mathbb{R}$ is called *of Calabi type* if for every displacable open subset $U \subset M$ the equality

$$f|_U = \mathrm{Cal}_U$$

holds.

In [7] the existence of a Calabi type quasimorphism was established for symplectic manifolds which are spherically monotone and the even part of whose quantum homology algebra is semisimple. We cannot explain these assumptions here, but refer the reader to the aforementioned article and the references therein for details. Here we can only sketch some ideas of the construction. The basic idea of Entov and Polterovich for constructing a Calabi type quasimorphism is to use the *spectral invariants* of G. For the present purpose it suffices to know that these are given by a map

$$c : QH_{ev}(M) \times G \to \mathbb{R}, \quad (a, g) \mapsto c(a, g),$$

where $QH_{ev}(M)$ is the even part of the quantum homology algebra of M. Then they prove the following result:

Lemma 4.5 (Entov-Polterovich). *If $QH_{ev}(M) = Q_1 \oplus \cdots \oplus Q_d$ denotes the decomposition of $QH_{ev}(M)$ into a direct sum of fields and e the unit of Q_1, then*

$$r := -c(e, \cdot) : G \to \mathbb{R}$$

is a continuous quasimorphism. Moreover,

   (i) *$r(gh) \geq r(g) + r(h)$ for all $g, h \in G$;*
   (ii) *If e_G denote the identity element of G, then $r(e_G) = 0$;*
   (iii) *r is conjugation-invariant.*

Up to a constant factor of $\mathrm{Vol}(M)$ *the homogeneization* $\widetilde{r}$ *of* r *is of Calabi type.*

For proofs see again [7], in particular Section 2.6, Theorem 3.1 and Proposition 3.3. We refer to r as the *spectral quasimorphism* on G. Based on the examples from finite-dimensional Lie groups it is reasonable to ask whether $\widetilde{r}$ sandwiches a partial order. For the study of this problem, we suggest the following terminology:

Definition 4.6. A closed symplectic manifold (M, ω) is called *Calabi orderable* if there exists a partial order $\leq$ on G and a dominant $g \in G^{++}$ with respect to this ordering such that the relative growth $\gamma(g, \cdot)$ is a Calabi type quasimorphism. In this case, $\leq$ is called a *Calabi order* on G.

We will now provide criteria which guarantee Calabi orderability. We call the spectral quasimorphism r *non-degenerate* if it satisfies

$$r(g) = r(g^{-1}) = 0 \Rightarrow g = e_G$$

for all $g \in G$. In this situation, Lemma 4.5 yields immediately the following corollary:

Corollary 4.7. *Let* (M, ω) *be a spherically monotone closed symplectic manifold the even part of whose quantum homology algebra is semisimple and whose spectral quasimorphism is non-degenerate. Then*

  (i) *The set*
$$G^+ := \{g \in G \,|\, r(g^{-1}) \leq 0\}$$
  is a closed, conjugation invariant pointed submonoid of G *and thus defines a partial order* $\leq$ *on* G.
  (ii) *The homogeneization* $\widetilde{r}$ *(and hence the Calabi quasimorphism* $\widetilde{\mu} := \mathrm{Vol}(M) \cdot \widetilde{r}$*) sandwich* $\leq$.

We refer to the order $\leq$ from Proposition 4.7 as the *spectral order* on G. We briefly recall some conditions that guarantee the non-degeneracy of the spectral quasimorphism:

Definition 4.8. A closed symplectic manifold (M, ω) is called
  - *rational* if $\omega(\pi_2(M)) \subset \mathbb{R}$ is a discrete subset;
  - *strongly semipositive*, if there is no spherical homology class $A \in \pi_2(M)$ such that $\omega(A) > 0$ and $2 - n \leq c_1(A) < 0$.

Then we have:

Theorem 4.9. *Let* (M, ω) *be a spherically monotone closed symplectic manifold the even part of whose quantum homology algebra is semisimple. If M is rational and strongly semipositive, then it is Calabi orderable. More precisely, a Calabi order is given by the spectral order* $\leq$. *Moreover,* $\mathfrak{X}(G, \leq) \cong \mathbb{R}$.

Proof. By a result of Oh [20, Theorem A] the conditions on M ensure that the spectral quasimorphism is non-degenerate. Thus the Calabi type quasimorphism $\widetilde{\mu}$ of Entov and Polterovich sandwiches the spectral order and the result follows. $\square$

The theorem applies in particular to $\mathbb{CP}^n$ with the Fubini-Study form; in particular

$$\mathfrak{X}(\widetilde{\mathrm{Ham}}(\mathbb{CP}^n), \leq) \cong \mathbb{R}$$

is *not* infinite-dimensional.

4.4. Collapse of the order space in the absence of quasimorphisms.

We have seen examples of both finite- and infinite-dimensional ordered Lie groups for which the order space is much smaller than expected. This collapsing phenomenon could in both cases be tracked back to the existence of a certain homogeneous quasimorphisms and one might thus get the impression that homogeneous quasimorphisms are the only reason for a collapse of the order space. The following example shows that this is not the case and, in fact, that the order space can collapse even in the complete absence of quasimorphisms: Consider the standard embeddings

$$\mathrm{Sp}(2, \mathbb{R}) \subset \mathrm{Sp}(4, \mathbb{R}) \subset \mathrm{Sp}(6, \mathbb{R}) \subset \dots$$

induced from the embeddings

$$T^*\mathbb{R} \subset T^*\mathbb{R}^2 \subset T^*\mathbb{R}^3 \subset \dots$$

Let us abbreviate by G_n the universal covering of $\mathrm{Sp}(2n, \mathbb{R})$. Then we have a similar chain for the groups G_n. Each G_n carries a unique continuous admissible partial ordering (the maximal partial ordering in the notation of the last section) and we denote the associated order semigroup by G_n^+. We then define

$$G := \varinjlim G_n = \bigcup G_n, \quad G^+ := \bigcup G_n^+ \subset G.$$

It turns out that G^+ defines an admissible order $\leq$ on G with

$$G^{++} = \bigcup G_n^{++}.$$

We claim that

$$(14) \qquad\qquad\qquad \mathfrak{X}(G, \leq) \cong \mathbb{R}.$$

Indeed, let $J \in C_1^{++}$ be the element in the center of $\mathfrak{k}$ defining the complex strucure on the symmetric space and $L := \{\exp(tJ) \,|\, t > 0\} \subset G^{++}$. We denote by $[L]$ the image of L in $\mathfrak{X}(G, \leq)$. Clearly, $[L] \cong \mathbb{R}$. We claim that $\mathfrak{X}(G, \leq) = [L]$. Indeed, let $g \in G^{++}$ and choose $n \in \mathbb{N}$ such that $g \in G_n$. Since $\exp(J)$ and g are dominant in G_n we have both $\mu_{G_n}(\exp(J)) > 0$ and $\mu_{G_n}(g) > 0$. Thus, there exists $t > 0$ such that $\mu_{G_n}(\exp(tJ)) = \mu_{G_n}(g)$. We now consider $\exp(tJ)$ and g as elements of G_n^{++} and denote by $d_{G_n}(\exp(tJ), g)$ the corresponding pseudo-distance. By Corollary 4.2 we have

$$d_{G_n}(\exp(tJ), g) = 0.$$

This implies

$$d_G(\exp(tJ), g) = 0,$$

since the natural map $\mathfrak{X}(G_n, \leq) \to \mathfrak{X}(G, \leq)$ is contractive. This shows that $[g] = [\exp(tJ)] \in \mathfrak{X}(G, \leq)$. Since $\exp(tJ) \in L$ we have $[g] \in [L]$ as claimed. This establishes (14). On the other hand, applying the Kotschick swindle [15] to the diagonal $\widetilde{SL_2}(\mathbb{R})$-subgroups of G, we see immediately that every homogeneous quasimorphism f on G restricts to a homomorphism on G_1. Since G_1 is simple, this homomorphism is trivial. But the only

quasimorphism of G_n restricting trivially to G_1 is the trivial one, whence f must be trivial on every G_n, hence on G. This shows that G does not possess any non-trivial homogeneous quasimorphism.

4.5. **Open problems.** We have seen in various examples that suitable homogeneous quasimorphisms allow the explicit computation of relative growth and, consequently, order spaces for ordered Lie groups. There are various directions into which our results can be extended. As far as finite-dimensional Lie groups are concerned, we have dealt with the extremal cases of simple and abelian Lie groups. In view of the structure theory of finite-dimensional Lie groups, the next step towards a complete understanding of order spaces would be to understand the behaviour of relative growth under semidirect products. For non-semisimple Lie groups the order space will probably not collapse, since the quasimorphism becomes trivial on the radical, whence it would be interesting to compute its precise form.

A second direction to be pursued is obviously the study of infinite-dimensional Lie groups. Here the interest is probably not in maximal generality, but rather in concrete computations of relative growth for specific classes of groups arising in contact and symplectic geometry. In the finite-dimensional case, a key step towards our computations was the reduction of continuous orders to invariant cones inside the Lie algebra. It would be interesting to know, whether such a reduction can also be used in the infinite-dimensional context.

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